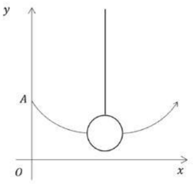
Q1a

A crane swings a wrecking ball along a two-dimensional path defined by the parametric equations

$$x = 10t \qquad y = 4.9t^2 - 4.9t + 2 \qquad 0 \le t \le 1$$

as shown in the diagram below.



x and y are, respectively, the horizontal and vertical displacements in metres from the origin, O, and t is the time in seconds. Point A indicates the initial position of the wrecking ball.

(a) (i) Write down the height of the wrecking ball when it is at point A.

(ii) Find the shortest distance between the wrecking ball and the ground during its motion.

[4]

(b) The destruction of a building requires the wrecking ball to strike it at a height of 1.4 m whilst on the upward part of its path.
Find the horizontal distance from point A at which the ball hits the building.

[4]

a) (i) When $t = 0$, $y = 4.9(0)^2 - 4.9(0) + 2 = 2$.

$\boxed{\text{At point A, the wrecking ball is at a height of 2 metres.}}$

(ii) $y = 4.9t^2 - 4.9t + 2$
$= 4.9(t^2 - t) + 2$
$= 4.9\left(\left(t - \frac{1}{2}\right)^2 - \frac{1}{4}\right) + 2$
$= 4.9\left(t - \frac{1}{2}\right)^2 - 1.225 + 2$
$y = 4.9\left(t - \frac{1}{2}\right)^2 + 0.775$

complete the square

y has a minimum of 0.775 when $t = \frac{1}{2}$

$\boxed{\text{The shortest distance between the wrecking ball and the ground during its motion is } 0.775\text{ m}}$

You could also consider $\frac{dy}{dt} = 0$ to find the value of t corresponding to the minimum value of y.

Q1B

From part (a) working, $y = 4.9\left(t - \frac{1}{2}\right)^2 + 0.775$

b) When $y = 1.4$

$4.9\left(t - \frac{1}{2}\right)^2 + 0.775 = 1.4$

$4.9\left(t - \frac{1}{2}\right)^2 = 0.625$

$\left(t - \frac{1}{2}\right)^2 = \frac{0.625}{4.9} = \frac{6.25}{49}$

$t - \frac{1}{2} = \pm\frac{2.5}{7} = \pm\frac{5}{14}$

$t = \frac{1}{2} \pm \frac{5}{14} \Rightarrow t = \frac{1}{7}, \frac{6}{7}$

But for the upward part of the path we must choose $t = \frac{6}{7}$

When $t = \frac{6}{7}$, $x = 10\left(\frac{6}{7}\right) = \frac{60}{7}$

$\boxed{\text{When the ball hits the building, the horizontal distance from point A is } \frac{60}{7} \text{ metres.}}$

Q2a

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

a) $y = 4\sin\theta \Rightarrow \frac{dy}{d\theta} = 4\cos\theta$

$x = 2\cos\left(\theta + \frac{\pi}{3}\right) \Rightarrow \frac{dx}{d\theta} = -2\sin\left(\theta + \frac{\pi}{3}\right)$

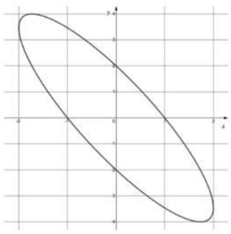
$$\frac{dy}{dx} = \frac{4\cos\theta}{-2\sin\left(\theta + \frac{\pi}{3}\right)}$$

$$\boxed{\frac{dy}{dx} = \frac{-2\cos\theta}{\sin\left(\theta + \frac{\pi}{3}\right)}}$$

Q2B

The graph of the ellipse E shown below is defined by the parametric equations

$x = 2\cos\left(\theta + \frac{\pi}{3}\right)$ $\quad y = 4\sin\theta$ $\quad -\pi \le \theta \le \pi$



(a) Find an expression for $\frac{dy}{dx}$ in terms of θ.

[3]

(b) Find the equation of the tangent to E, at the point where $\theta = -\frac{\pi}{6}$, giving your answer in the form $y = a - bx$, where a and b are real numbers that should be given in exact form.

[4]

Equation of a line with gradient m through (x_1, y_1) is
$y - y_1 = m(x - x_1)$

From part (a), $\frac{dy}{dx} = \frac{-2\cos\theta}{\sin\left(\theta + \frac{\pi}{3}\right)}$

b) when $\theta = -\frac{\pi}{6}$,

$x = 2\cos\left(\frac{\pi}{6}\right) = \sqrt{3}$ $\quad y = 4\sin\left(-\frac{\pi}{6}\right) = -2$

$\frac{dy}{dx} = -\frac{2\cos(-\pi/6)}{\sin(\pi/6)} = -\frac{\sqrt{3}}{1/2} = -2\sqrt{3}$

The equation of the tangent is

$y - (-2) = -2\sqrt{3}(x - \sqrt{3})$

$y + 2 = -2\sqrt{3}x + 6$

$$\boxed{y = 4 - 2\sqrt{3}x}$$

Q3

The curve C has parametric equations

$$x = 3t \qquad y = t + \frac{1}{t} \qquad t > 0$$

Find the equation of the normal to C at the point where C intersects the line $y = x$.

[9]

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

gradient of normal $= -\frac{1}{dy/dx}$

Equation of a line with gradient m through (x_1, y_1) is $y - y_1 = m(x - x_1)$

C intersects the line $y = x$ when

$\underbrace{3t = t + \frac{1}{t}}_{x=y} \Rightarrow 3t^2 = t^2 + 1 \Rightarrow 2t^2 = 1 \Rightarrow t = \pm\frac{1}{\sqrt{2}}$

But $t > 0$, so $t = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

When $t = \frac{\sqrt{2}}{2}$

$x = 3\left(\frac{\sqrt{2}}{2}\right) = \frac{3\sqrt{2}}{2} \qquad y = \frac{\sqrt{2}}{2} + \frac{1}{\sqrt{2}/2} = \frac{\sqrt{2}}{2} + \sqrt{2} = \frac{3\sqrt{2}}{2}$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1 - \frac{1}{t^2}}{3} = \frac{1 - \frac{1}{(\sqrt{2}/2)^2}}{3} = -\frac{1}{3}$

The gradient of the normal is $-\frac{1}{(-1/3)} = 3$

The equation of the normal is

$y - \frac{3\sqrt{2}}{2} = 3\left(x - \frac{3\sqrt{2}}{2}\right)$

$y - \frac{3\sqrt{2}}{2} = 3x - \frac{9\sqrt{2}}{2}$

$y = 3x - \frac{9\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}$

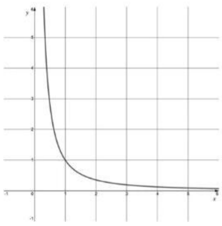
$\boxed{y = 3x - 3\sqrt{2}}$

Q4

The graph of the curve defined by the parametric equations

$$x = e^{2t} \qquad y = e^{-3t}$$

is shown below.



(i) Verify that the graph passes through the point $(1, 1)$.

(ii) Prove that the line with equation $y = x$ is **not** the normal to the curve at the point $(1, 1)$.

[6]

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

gradient of normal $= -\frac{1}{dy/dx}$

(i) When $t = 0$, $x = e^0 = 1$ and $y = e^0 = 1$

So the curve passes through $(1, 1)$

(ii) $\frac{dx}{dt} = 2e^{2t} \qquad \frac{dy}{dt} = -3e^{-3t}$

$\frac{dy}{dx} = \frac{-3e^{-3t}}{2e^{2t}} = -\frac{3}{2}e^{-5t}$

So at $(1, 1)$ where $t = 0$,

$\frac{dy}{dx} = -\frac{3}{2}e^0 = -\frac{3}{2}$

So the gradient of the normal at $(1, 1)$ is

$-\frac{1}{dy/dx} = -\frac{1}{(-3/2)} = \frac{2}{3}$

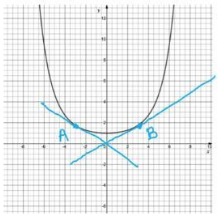
The gradient of $y = x$ is 1.

Therefore $y = x$ is not the normal to the curve at the point $(1, 1)$

Q5a

The diagram below shows a sketch of the curve defined by the parametric equations

$x = 4t \qquad y = e^{t^2}$



$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$

All lines through the origin are of the form $y = mx$

The tangents to the curve that pass through the origin meet the curve at points A and B

(a) Show that the values of t at points A and B are $t = -\frac{\sqrt{2}}{2}$ and $t = \frac{\sqrt{2}}{2}$.

[5]

(b) Hence, or otherwise, show that the area of the triangle OAB is $2\sqrt{2}\,e^{\frac{1}{2}}$ square units.

[3]

a) The gradients of the tangents to the curve are given by

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2te^{t^2}}{4} = \frac{1}{2}te^{t^2}$$

Lines through the origin with that gradient are of the form $y = \left(\frac{1}{2}te^{t^2}\right)x$

Where the tangents touch the curve, $x = 4t$ and $y = e^{t^2}$. Therefore:

$$e^{t^2} = \left(\tfrac{1}{2}te^{t^2}\right)(4t)$$

$$e^{t^2} = 2t^2e^{t^2}$$

$$1 = 2t^2$$

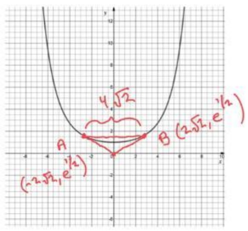
$$t^2 = \frac{1}{2} \Rightarrow t = \sqrt{\frac{1}{2}}$$

$$t = \frac{\sqrt{2}}{2} \text{ or } -\frac{\sqrt{2}}{2}$$

Q5b

The diagram below shows a sketch of the curve defined by the parametric equations

$x = 4t \qquad y = e^{t^2}$



The tangents to the curve that pass through the origin meet the curve at points A and B

(a) Show that the values of t at points A and B are $t = -\frac{\sqrt{2}}{2}$ and $t = \frac{\sqrt{2}}{2}$.

[5]

(b) Hence, or otherwise, show that the area of the triangle OAB is $2\sqrt{2}\,e^{\frac{1}{2}}$ square units.

b) When $t = -\frac{\sqrt{2}}{2}$: $x = 4\left(-\frac{\sqrt{2}}{2}\right) = -2\sqrt{2}$

$y = e^{\left(-\frac{\sqrt{2}}{2}\right)^2} = e^{1/2}$

When $t = \frac{\sqrt{2}}{2}$: $x = 4\left(\frac{\sqrt{2}}{2}\right) = 2\sqrt{2}$

$y = e^{\left(\frac{\sqrt{2}}{2}\right)^2} = e^{1/2}$

The area of triangle OAB is

$$\frac{1}{2} \times (4\sqrt{2}) \times \left(e^{\frac{1}{2}}\right)$$

$$= 2\sqrt{2}\,e^{\frac{1}{2}}$$